Tentamen Numerical Mathematics 2
March 29, 2016

Duration: 3 hours.
In front of the questions one finds the weights used to determine the final mark.

Problem 1

a. Consider the problem $F(x) = d$ with $F(x) = Ax$.

(i) [2] Give the definition of the absolute condition number of this problem and express it in terms of A, x and d for this case.

(ii) [1] Give the definition of the relative condition number of this problem and express it in terms of A, x and d for this case.

(iii) [1] How is the expression of the previous part related to the condition number of A.

b. Consider a system $Ax = b$ where $b = [1, 1]^T$. For which we consider two matrices

$$A_1 = \begin{bmatrix} 2.001 & 2 \\ 2 & 2.001 \end{bmatrix}, \text{ and } A_2 = \begin{bmatrix} 2.001 & -2 \\ -2 & 2.001 \end{bmatrix}$$

(i) [2] Show that A_1 and A_2 have the same eigenvalues and eigenvectors.

(ii) [1] Show that the 2-norms of A_1 and A_2 are equal and moreover also the 2-norms of inverses of the two matrices are equal and hence that $\kappa_2(A_1) = \kappa_2(A_2)$.

(iii) [2] Show that the relative condition number as defined in item a.(ii) is different. Which of the two solutions will suffer most from round-off error propagation?

c. Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

(i) [3] Using Gerschgorin's theorems, localize the eigenvalues of this matrix and show that the matrix is positive definite.

(ii) [1] Is pivoting needed to solve a system $Ax = b$ with this matrix?

(iii) [1] Which direct method do you advice to use for this system and why?

d. Suppose we solve the problem $Ax = b$ with Gaussian elimination with pivoting.

(i) [1] Explain what one tries to bound with the backward error for this problem.

(ii) [1] What does one want to bound with the forward error for this problem?

(iii) [1] How is the backward error related to the forward error?

Problem 2

a. [3] Let A and E be real symmetric matrices and let μ be the eigenvalue of $A + E$. Show that

$$\min_{\lambda \in \sigma(A)} |\lambda - \mu| \leq ||E||_2$$

b. Consider the iteration

$$(A - \mu I)y_{n+1} = x_n, \quad x_{n+1} = y_{n+1}/||y_{n+1}||, \quad n = 0, 1, 2, ...$$

where x_0, A and μ are given.

(i) [2] Where does the vector x_n converge to when A is real symmetric and its eigenvalues are all different?

(ii) [2] What determines the speed of convergence of the iteration?

c. Suppose we have a full real symmetric matrix A. As a preprocessing step for the QR-method A is brought to tridiagonal form by an orthogonal similarity transformation.

(i) [1] Why is one doing this?

(ii) [3] Bring the following matrix to tridiagonal form using a Householder matrix:

$$A = \begin{bmatrix} 6 & 4 & 3 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

d. Consider the QR-method used to compute all the eigenvalues of a real symmetric matrix A.

(i) [1] Give the basic step of the QR-method.

(ii) [1] Give the basic step of the QR-method including a shift.

(iii) [1] How is the shift used to speed up the convergence?

(iv) [1] Does a shift effect the order of the eigenvalues on the diagonal of the iterate?

Problem 3

a. Consider the approximation of a function $f(x)$ defined on [-1,1] by a Chebyshev expansion $C_n(x) = \sum_{i=0}^{n} c_i T_i(x)$.

(i) [3] Give the expression for c_i following from a Least Squares minimization.

(ii) [2] Explain why the solution in the previous part is the preferred approximation to the polynomial minimax approximation of $f(x)$.

(iii) [1] Consider the function $f(x)$ which is equal to $1+x$ on [-1,0] and $1-x$ on [0,1]. Will there be pointwise convergence of $C_n(x)$ for $n \to \infty$?

b. Consider the integral $\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x)dx$

(i) [2] Show that the zeros of the n-th degree Chebyshev polynomial are given by $x_k = \cos((k - \frac{1}{2})\pi/n)$ for $k = 1, ...n$.

(ii) [3] Show that if the x_k are picked as interpolation points that the rule is exact for polynomials of degree $2n - 1$.

(iii) [1] Show that the above integral is equal to the integral $\int_0^\pi f(\cos(\theta))d\theta$.

(iv) [1] It can be shown that the application of the Gauss-Chebyshev rule to the integral above is the same as applying the composite midpoint rule to $\int_0^\pi f(\cos(\theta))d\theta$. What does this mean for the rate of convergence if we integrate $\exp(\cos(\theta)))$ over the interval $[0, \pi]$ with the composite midpoint rule?